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Periodic and aperiodic solitary wave solutions of the non-linear Klein–Gordon equation without dispersion

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Abstract. The Klein–Gordon equation without dispersion, and with quadratic and cubic non-linearities, has been studied in one and higher dimensions. Algebraic solitary wave solutions in all cases, as well as higher-order modes in higher dimensions (similar to non-linear optics) have been shown to exist corresponding to specific initial values. While in the one-dimensional case, arbitrary initial values yield periodic solutions, asymptotically stable solutions are shown to exist in the higher-dimensional case. For both one- and higher-dimensional cases, solutions tending to zero with distance are shown to be achieved for other initial conditions by incorporating a small amount of 'saturating' fourth-order non-linearity. Finally, it is shown how a general Klein–Gordon equation with dispersion and a forcing term may be reduced to the equation discussed in the paper.

1. Introduction

The Klein–Gordon equation plays a fundamental role as a model equation in non-linear field theories (Bjorken and Drell 1964, Hobart 1963), in lattice dynamics (Scott 1969, Bishop and Schneider 1978) and in non-linear optics (Chiao *et al* 1964, Haus 1966). Stationary baseband solutions of the equation come about as a balance between the non-linearity and the dispersion and thus represent solitary wave solutions to the system. While analytic solutions in powers of sech functions can be determined in one dimension (Korpel 1979), radially symmetric higher-dimensional solutions have no simple analytic form; these solutions are thus obtained using numerical methods (Chiao *et al* 1964, Haus 1966) or by a variational technique (Small 1972). Envelope solutions can also be determined by first showing that the PDE for the envelope satisfies the same stationary baseband Klein–Gordon equation with suitably modified coefficients (Korpel and Banerjee 1984).

Several non-linear generalisations of the Klein–Gordon equation also have exact analytic solutions, which in special cases are solitary waves (Burt and Reid 1976, Burt 1978, 1980). In most cases the solutions are in terms of exponentials; however, in at least one special case, algebraic-type solitary waves have been reported (Burt 1978). Klein–Gordon equations having a constant forcing term and damping have been studied by Lal (1985, 1986) and envelope Klein–Gordon systems in one and higher dimensions have been analysed on the basis of similarity transformations (Tajiri 1984a) and a reduction to the second Painlevé equation (Tajiri 1984b) with a view to determining N-soliton solutions (Tajiri 1984c).

As noted by Burt (1980), solutions of certain non-linear generalisations of the Klein-Gordon equation exhibit soliton properties even when the dispersion vanishes. No rigorous physical explanation for this seems to be available, though intuitively speaking we might argue that, for instance, a quadratic non-linearity can balance a cubic non-linearity in the following way. Consider, for instance, the kinematic wave equation of the form (Whitham 1974)

$$\partial\psi/\partial t + c_0(1+\beta_2\psi+\beta_3\psi^2)\partial\psi/\partial x = 0 \tag{1}$$

where ψ represents the wavefunction, c_0 is the (linear) phase velocity and β_2, β_3 denote the quadratic and cubic non-linearity coefficients. Consider furthermore, the case where $\beta_2 > 0$, $\beta_3 < 0$ and where ψ at $t = 0$ is a baseband signal greater than zero. Then, with time, the leading edge of the signal steepens while the trailing edge smoothens under the action of the quadratic non-linearity alone, while the reverse occurs under the effect of the cubic non-linearity. The combined effect can be visualised as a balancing process whereby the signal may finally evolve into a shape which remains unchanged during propagation.

The organisation of this paper is as follows. In § 2, we derive (algebraic) solitary wave solutions of the non-linear Klein-Gordon equation (having quadratic and cubic non-linearities) without the dispersion term, in one and higher dimensions. Now conventional Klein-Gordon systems (i.e. with dispersion) also exhibit periodic solutions expressible in terms of elliptic integrals. We have found that the non-linear Klein-Gordon equation without dispersion and in one dimension also exhibits a similar property, for arbitrary initial values (except the initial value corresponding to the aperiodic solution and trivial cases). This is intensively discussed in § 3. Examples corresponding to typical initial values are also provided and plotted. Furthermore, non-linear optics solutions (Haus 1966) also predict the existence of higher-order modes that decay to zero for higher-dimensional propagation. In the case of the Klein-Gordon equation under consideration, we have checked that this is true, by employing a numerical scheme. For arbitrary initial conditions not corresponding to any one of these modes, the solutions no longer decay to zero (as in the non-linear optics case), however, we observe decaying oscillations with decreasing periods that tend to a limiting value. This is presented in § 4. Finally, in § 5, we show how a Klein-Gordon equation having dispersion as well as the quadratic and cubic non-linearities and a constant forcing term of the type mentioned by Lal (1985, 1986) can be reduced to the Klein-Gordon equation we consider, and thus establish aperiodic (algebraic) and periodic solutions to more general systems.

2. Algebraic solitary wave solutions of the non-linear Klein-Gordon equation without dispersion

We will consider, in this paper, the non-linear Klein-Gordon equation without dispersion of the form

$$\partial^2\psi/\partial t^2 - c_0^2\nabla^2\psi = A_2\psi^2 + A_3\psi^3 \qquad \psi \triangleq \psi(x, y, z, t) \qquad \nabla^2 \triangleq \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2 \tag{2}$$

where c_0 represents the (linear) phase velocity and where A_2, A_3 denote the quadratic and cubic non-linearity coefficients. In our quest for the solitary wave solution, we now introduce a travelling frame of reference

$$\tilde{\xi} = x - vt \tag{3}$$

where v is the anticipated velocity of the solitary wave. Substituting (3) in (2) and defining

$$\xi = \frac{1}{\sqrt{c_0^2 - v^2}}\,\tilde{\xi} \qquad \eta = \frac{1}{c_0}\,y \qquad \zeta = \frac{1}{c_0}\,z \tag{4}$$

we obtain

$$\begin{aligned} &\frac{\partial^2 \tilde{\psi}}{\partial \xi^2} + \frac{\partial^2 \tilde{\psi}}{\partial \eta^2} + \frac{\partial^2 \tilde{\psi}}{\partial \zeta^2} \triangleq \tilde{\nabla}^2 \tilde{\psi} = -A_2 \tilde{\psi}^2 - A_3 \tilde{\psi}^3 \\ &\tilde{\psi} \triangleq \tilde{\psi}(\xi, \eta, \zeta) = \psi(x, y, z, t). \end{aligned} \tag{5}$$

Assuming radial symmetry (if we are working in higher dimensions), (5) simplifies to

$$\frac{d^2 \tilde{\tilde{\psi}}}{dr^2} + \frac{\tilde{n}}{r}\frac{d\tilde{\tilde{\psi}}}{dr} = -A_2 \tilde{\tilde{\psi}}^2 - A_3 \tilde{\tilde{\psi}}^3 \qquad \tilde{\tilde{\psi}} \triangleq \tilde{\tilde{\psi}}(r) = \tilde{\psi}(\xi, \eta, \zeta) \tag{6}$$

where $\tilde{n} = 0, 1, 2$ in the one-dimensional case, circular and spherical symmetries respectively. Finally, normalisation using

$$\Psi = -\frac{A_3}{A_2}\,\tilde{\tilde{\psi}} \qquad R = \left(\frac{A_2^2}{A_3}\right)^{1/2} r \qquad A_3 > 0 \tag{7}$$

reduces (6) to

$$\frac{d^2 \Psi}{dR^2} + \frac{\tilde{n}}{R}\frac{d\Psi}{dR} = \Psi^2 - \Psi^3 \qquad \Psi \triangleq \Psi(R). \tag{8}$$

Lemma 2.1. Rational algebraic solutions to (8) finite at $R = 0$ and ∞ are of the form

$$\Psi(R) = \frac{a}{1 + bR^2} \qquad a, b \in \mathbb{R}. \tag{9}$$

Proof. Changing R to $-R$ in (8) leaves the equation invariant, which means $\Psi(-R) = \Psi(R)$. Assume, therefore, a rational algebraic form for Ψ as

$$\Psi(R) = \frac{P_{2n}(R)}{Q_{2n+2m}(R)} \qquad n,\ n + m \in Z \text{ and finite} \tag{10}$$

where

$$P_{2n}(R) \triangleq a_0 + a_1 R^2 + A_2 R^4 + \ldots + a_n R^{2n}$$

and

$$Q_{2n+2m}(R) \triangleq b_0 + b_1 R^2 + b_2 R^4 + \ldots + b_{n+m} R^{2(n+m)}. \tag{11}$$

Then

$$\begin{aligned} \frac{d\Psi(R)}{dR} &= \frac{P'_{2n}(R) Q_{2n+2m}(R) - P_{2n}(R) Q'_{2n+2m}(R)}{Q^2_{2n+2m}(R)} \\ &\triangleq P^{(1)}_{4n+2m-1}(R) / Q^{(1)}_{4n+4m}(R) \triangleq P^{(2)}_{6n+4m-1}(R) / Q^{(2)}_{6n+6m}(R). \end{aligned} \tag{12}$$

The superscripts on P and Q have merely been introduced for differentiating them from the original P and Q defined in (11); the subscripts are important and denote the orders of the polynomials. Similarly,

$$\frac{d^2\Psi(R)}{dR^2} \triangleq P^{(3)}_{6n+4m-2}(R)/Q^{(2)}_{6n+6m}(R) \tag{13}$$

$$\Psi^2 \triangleq P^{(4)}_{6n+2m}(R)/Q^{(2)}_{6n+6m}(R) \tag{14}$$

$$\Psi^3 \triangleq P^{(5)}_{6n}(R)/Q^{(2)}_{6n+6m}(R). \tag{15}$$

Substituting (12)-(15) in (8) and comparing the orders of the numerator polynomials,

$$6n+4m-2=6n+2m$$

implies that

$$m=1 \tag{16}$$

i.e. the order of the denominator in (10) should be higher than the order of the numerator by 2. Note that this also ensures that Ψ is finite ($=0$) as $R\to\infty$.

Furthermore, inspection of (10) reveals that $(n+1)+(n+m)$ unknown coefficients have to be solved for, since b_0 can be set equal to 1 without loss of generality. With $m=1$, this means that the number of unknowns is $2n+2$. The total number of available equations is $(6n+2)/2+1=3n+2$, since $6n+2$ is the order of the numerator polynomials. In order to have non-trivial solutions, we must have

$$2n+2 \geqslant 3n+2$$

implying

$$n=0.$$

Finally, putting $a_0=a$; $b_1=b$, equation (9) follows.

Substituting (9) in (8), and evaluating a and b, $\Psi(R)$ can be expressed as

$$\Psi(R)=\frac{4}{(3-\tilde{n})}\left(1+\frac{2}{(3-\tilde{n})^2}R^2\right)^{-1} \qquad \tilde{n}=0,1,2. \tag{17}$$

As a check, for $\tilde{n}=2$ (spherically symmetric case), (17) yields the same solution as in Burt (1978). Also, the denormalised solution in terms of r can be written as

$$\tilde{\psi}(r)=[4/(3-\tilde{n})]\left(-\frac{A_2}{A_3}\right)\left[1+\frac{2}{(3-\tilde{n})^2}\left(\frac{A_2^2}{A_3}\right)r^2\right]^{-1} \qquad A_3>0. \tag{18}$$

Heuristically speaking, (18) makes sense since $A_2<0$, $A_3>0$ corresponds to non-linearities having opposite signs and can create the balance, referred to earlier, for positive baseband signals. It may be readily argued that when the non-linearities are of the same sign, the signal has to be negative to create the same balance. The solitary wave moves with a velocity $v<c_0$. The velocity may be related to the 'width' of the solitary wave in (18) through (4)-(6).

3. Periodic solutions of the non-linear Klein–Gordon equation without dispersion in one dimension

We will use, as our starting point, equation (8), which is a non-linear ODE in Ψ, with $\tilde{n}=0$

$$\mathrm{d}^2\Psi/\mathrm{d}R^2=\Psi^2-\Psi^3. \tag{19}$$

Clearly, $\Psi=0$ and 1 represent trivial solutions. Multiplying both sides of (19) by $\mathrm{d}\Psi/\mathrm{d}R$ and integrating with respect to R, we obtain

$$(\mathrm{d}\Psi/\mathrm{d}R)^2=\tfrac{2}{3}\Psi^3-\tfrac{1}{2}\Psi^4+K \tag{20}$$

where K is an integration constant.

Assume, now, that at $R=0$, $\Psi=\Psi_0$ and $\mathrm{d}\Psi/\mathrm{d}R=0$; this gives

$$K=\tfrac{1}{2}\Psi_0^4-\tfrac{2}{3}\Psi_0^3. \tag{21}$$

In order to reduce (20) to a tractable integral, we set

$$\Psi=\hat{\Psi}+\alpha \tag{22}$$

to obtain

$$(\mathrm{d}\hat{\Psi}/\mathrm{d}R)^2=(\tfrac{2}{3}\alpha^3-\tfrac{1}{2}\alpha^4+K)+(2\alpha^2-2\alpha^3)\hat{\Psi}+(2\alpha-3\alpha^2)\hat{\Psi}^2+(\tfrac{2}{3}-2\alpha)\hat{\Psi}^3-\tfrac{1}{2}\hat{\Psi}^4. \tag{23}$$

We then set the constant term equal to zero, yielding

$$\tfrac{2}{3}\alpha^3-\tfrac{1}{2}\alpha^4+K=0 \tag{24}$$

or, using (21),

$$\tfrac{2}{3}(\alpha-\Psi_0)(\alpha^2+\Psi_0\alpha+\Psi_0^2)=\tfrac{1}{2}(\alpha-\Psi_0)(\alpha^3+\alpha^2\Psi_0+\alpha\Psi_0^2+\Psi_0^3). \tag{25}$$

A value of α satisfying (25) is

$$\alpha=\alpha_1=\Psi_0. \tag{26}$$

To find other (real) value(s), we have to solve the cubic equation

$$\alpha^3+(\Psi_0-\tfrac{4}{3})\alpha^2+(\Psi_0-\tfrac{4}{3})\Psi_0\alpha+(\Psi_0-\tfrac{4}{3})\Psi_0^2=0. \tag{27}$$

Two cases need to be considered: (a) $\Psi_0=\frac{4}{3}$ and (b) $\Psi_0\neq\frac{4}{3}$. These are discussed separately in the two following subsections. As we shall subsequently show, case (a) yields the aperiodic solution discussed in § 2; however, we include the discussion for the sake of completeness.

3.1. *Case (a):* $\Psi_0=\frac{4}{3}$

The possible values of α are

$$\alpha_1=\tfrac{4}{3} \qquad \text{from (26)} \tag{28}$$

and

$$\alpha_2=0 \qquad \text{from (27)}. \tag{29}$$

We will discuss first the case of $\alpha=\alpha_2=0$ for which $\Psi=\hat{\Psi}$ (from (22)). Also, from (21), $K=0$; hence, from (23),

$$(\mathrm{d}\Psi/\mathrm{d}R)^2=\tfrac{2}{3}\Psi^3-\tfrac{1}{2}\Psi^4 \tag{30}$$

or, upon integration,

$$\pm\tfrac{1}{3}R\Psi=(\tfrac{2}{3}\Psi-\tfrac{1}{2}\Psi^2)^{1/2}+\text{constant}. \tag{31}$$

Employing the condition that $\Psi(R=0)=\Psi_0=\frac{4}{3}$, the value of the constant becomes zero; hence, from (31), we obtain

$$\Psi(R)=\tfrac{4}{3}(1+\tfrac{2}{9}R^2)^{-1} \tag{32}$$

in agreement with (17) for $\tilde{n}=0$.

For $\alpha=\alpha_1=\Psi_0=\frac{4}{3}$, $K=0$ once again, and we recover (32).

Hence, for both values of α, we obtain the same aperiodic solution.

3.2. Case (b): $\Psi_0\neq\frac{4}{3}$

In this case, there are four possible values of α to examine, one given by (26) and the other three given by the roots of the cubic equation (27). In what follows, we will go through a detailed analysis of this case to expose in depth the relationships between different possible solutions.

Lemma 3.1. Of the three roots of (27), one is real and the other two are complex conjugates.

Proof. In (27), setting

$$\alpha=\gamma-\tfrac{1}{3}(\Psi_0-\tfrac{4}{3}) \tag{33}$$

to eliminate the quadratic power of the unknown, we get, after some algebra,

$$\gamma^3-q\gamma-r=0 \tag{34}$$

with

$$q=-[\tfrac{2}{3}\Psi_0+\tfrac{4}{9}](\Psi_0-\tfrac{4}{3}) \tag{35}$$

and

$$r=-[\tfrac{20}{27}\Psi_0^2+\tfrac{20}{81}\Psi_0+\tfrac{32}{243}](\Psi_0-\tfrac{4}{3}). \tag{36}$$

Now, following Pipes and Harvill (1970), (34) will have a real root and two complex (conjugate) roots if

$$27r^2>4q^3. \tag{37}$$

But (37) with (35) and (36) may readily reduce to

$$\Psi_0^2+\tfrac{2}{3}\Psi_0+\tfrac{1}{3}>0 \tag{38}$$

which is always true, and the lemma is proved.

From the results of the above lemma, it suffices to consider the two real roots of $\alpha(=\alpha_{1,2})$ for each value of Ψ_0. Now, from (23), without the constant term, we have

$$\left(\frac{\mathrm{d}\hat{\Psi}}{\mathrm{d}R}\right)^2=(2\alpha^2-2\alpha^3)\hat{\Psi}+(2\alpha-3\alpha^2)\hat{\Psi}^2+(\tfrac{2}{3}-2\alpha)\hat{\Psi}^3-\tfrac{1}{2}\hat{\Psi}^4. \tag{39}$$

Putting

$$\hat{\Psi}=1/\phi \tag{40}$$

(39) becomes

$$\left(\frac{\mathrm{d}\phi}{\mathrm{d}R}\right)^2 = a\phi^3 + b\phi^2 + c\phi + d \triangleq p(\phi) \tag{41}$$

where

$$a = 2\alpha^2 - 2\alpha^3 \tag{42a}$$

$$b = 2\alpha - 3\alpha^2 \tag{42b}$$

$$c = \tfrac{2}{3} - 2\alpha \tag{42c}$$

$$d = -\tfrac{1}{2}. \tag{42d}$$

The solution to ϕ, and hence to $\hat{\Psi}$ and Ψ, can be expressed in terms of Jacobian elliptic functions (Abramowitz and Stegun 1970). Hence, Ψ is periodic in nature. With $\mathrm{d}\Psi(0)/\mathrm{d}R = 0$, if $\Psi_0 > 1$ (<1), it must correspond to the maximum (minimum) value of Ψ. This is obvious if we examine (19). We will call the maximum value Ψ_{M} and the minimum value Ψ_{m} with Ψ_{ext} denoting either of the two.

We may see by comparison that (24) for α and (21) for Ψ_{ext} are the same. Hence both Ψ_{M} and Ψ_{m} satisfy (24) with α replaced by $\Psi_{\mathrm{M,m}}$. Furthermore, since we have shown that (24) has two possible real solutions, they must be Ψ_{M} and Ψ_{m}. Thus if $\Psi_{\mathrm{M(m)}}$ is the given initial value, $\Psi_{\mathrm{m(M)}}$ will be given by the solution to (27).

Lemma 3.2. If $1 < \Psi_{\mathrm{M}} < \frac{4}{3}$, then $0 < \Psi_{\mathrm{m}} < 1$; and if $\Psi_{\mathrm{M}} > \frac{4}{3}$, then $\Psi_{\mathrm{m}} < 0$.

Proof. On the basis of the argument preceding the lemma, if the initial value is Ψ_{M}, the equation satisfied by Ψ_{m} is

$$\tilde{p}(\Psi_{\mathrm{m}}) \triangleq \Psi_{\mathrm{m}}^3 + (\Psi_{\mathrm{M}} - \tfrac{4}{3})\Psi_{\mathrm{m}}^2 + (\Psi_{\mathrm{M}} - \tfrac{4}{3})\Psi_{\mathrm{M}}\Psi_{\mathrm{m}} + (\Psi_{\mathrm{M}} - \tfrac{4}{3})\Psi_{\mathrm{M}}^2 = 0. \tag{43}$$

If $\Psi_{\mathrm{M}} < \frac{4}{3}$, putting $\Psi_{\mathrm{m}} = 0$ in the expression for $\tilde{p}(\Psi_{\mathrm{m}})$ yields $\tilde{p}(0) < 0$. Also, putting $\Psi_{\mathrm{m}} = 1$ in $\tilde{p}(\Psi_{\mathrm{m}})$ gives

$$\begin{aligned}\tilde{p}(1) &= (\Psi_{\mathrm{M}} - 1)[(\Psi_{\mathrm{M}} + \tfrac{1}{3})^2 + \tfrac{2}{9}] \\ &> 0\end{aligned}$$

since $\Psi_{\mathrm{M}} > 1$. Hence, $0 < \Psi_{\mathrm{m}} < 1$.

Similarly, if $\Psi_{\mathrm{M}} > \frac{4}{3}$, $\tilde{p}(0) > 0$, and $\tilde{p}(-\infty) = -\infty$; hence $\Psi_{\mathrm{m}} < 0$.

Lemma 3.3. The polynomial $p(\phi)$ defined in (41) has only one real root

$$\beta = 1/(\Psi_{\mathrm{ext}} - \alpha) \tag{44}$$

where $\alpha = \Psi_{\mathrm{M(m)}}$ if $\Psi_{\mathrm{ext}} = \Psi_{\mathrm{m(M)}}$.

Proof. From (41),

$$\begin{aligned}p(\phi) &= a\left(\phi^3 + \frac{b}{a}\phi^2 + \frac{c}{a}\phi + \frac{d}{a}\right) \\ &= a(\tau^3 - \tilde{q}\tau - \tilde{r}) \qquad \tau \triangleq \phi + \tfrac{1}{3}b/a\end{aligned} \tag{45}$$

where

$$\tilde{q} = -(\alpha - \tfrac{4}{3})\frac{\alpha^3}{(2\alpha^3 - 2\alpha^2)^3} \tag{46}$$

$$\tilde{r} = \tfrac{2}{9}(\alpha - \tfrac{4}{3})\frac{\alpha^3}{(2\alpha^3 - 2\alpha^2)^3} \tag{47}$$

upon using (42). As in lemma 3.1, $p(\phi)$ will have one real and two complex (conjugate) roots if

$$27\tilde{r}^2 > 4\tilde{q}^3 \tag{48}$$

i.e. if

$$4(\alpha - 1)^2[(\alpha + \tfrac{1}{3})^2 + \tfrac{2}{9}] > 0 \tag{49}$$

which is always true, and hence $p(\phi)$ has only one real root.

To show that the real root is $\beta = 1/(\Psi_{\text{ext}} - \alpha)$, we substitute in (41) to get

$$\begin{aligned} p(\beta) &= p[1/(\Psi_{\text{ext}} - \alpha)] \\ &= [a + b(\Psi_{\text{ext}} - \alpha) + c(\Psi_{\text{ext}} - \alpha)^2 + d(\Psi_{\text{ext}} - \alpha)^3]/(\Psi_{\text{ext}} - \alpha)^3. \end{aligned} \tag{50}$$

Now, in (25), if $\Psi_0 = \Psi_{\text{ext}} = \Psi_{\text{M}}(\Psi_{\text{m}})$, then for $\alpha \neq \alpha_1$, (27) should give the other value of $\alpha(=\Psi_{\text{m}}(\Psi_{\text{M}}))$. From (50), the term in square brackets can be re-expressed, using (42) as:

$$\begin{aligned} &(2\alpha^2 - 2\alpha^3) + (2\alpha - 3\alpha^2)(\Psi_{\text{ext}} - \alpha) + (\tfrac{2}{3} - 2\alpha)(\Psi_{\text{ext}} - \alpha)^2 - \tfrac{1}{2}(\Psi_{\text{ext}} - \alpha)^3 \\ &\qquad = -\tfrac{1}{2}[\alpha^3 + (\Psi_{\text{ext}} - \tfrac{4}{3})\alpha^2 + (\Psi_{\text{ext}} - \tfrac{4}{3})\Psi_{\text{ext}}\alpha + (\Psi_{\text{ext}} - \tfrac{4}{3})\Psi_{\text{ext}}^2] \\ &\qquad = 0 \end{aligned}$$

using (27), and the lemma is proved.

At this point, let us briefly summarise the results proved thus far for the benefit of readers. In looking for solutions to the ODE in (19), we have shown that besides the trivial solutions, algebraic solutions can be obtained for an initial condition $\Psi_0 = \frac{4}{3}$. For $\Psi_0 \neq \frac{4}{3}$, the solutions are periodic in nature and expressible in terms of elliptic integrals. If $1 < \Psi_0 < \frac{4}{3}$, it corresponds to the maximum value of the periodic function, with the minimum value lying between 0 and 1. If $\Psi_0 > \frac{4}{3}$, it is once again the maximum value with the minimum being less than 0. Similarly, for $\Psi_0 < 1$, it corresponds to the minimum value, with the maximum value lying between 1 and $\frac{4}{3}$ if $\Psi_0 > 0$, and with the maximum value greater than $\frac{4}{3}$ if $\Psi_0 < 0$. Given a certain initial condition, the corresponding minimum or maximum can be determined through solving (43) which, in fact, has the same structure as (27). Furthermore, corresponding to a given initial condition, there can be two possible values for α, one equal to the initial condition and the other given by (27), which is equal to the other extremum for the given initial condition. Finally, we have shown that the root of the polynomial $p(\phi)$ in (41), which is pertinent in defining the properties of the periodic solution, is given by (44).

The summary above exposes four different cases to be considered. Corresponding to a given initial condition $\Psi_0 = \Psi_{\text{ext}}(=\Psi_{\text{M(m)}})$, there are two values of α; $\alpha_1 > 1$ and $\alpha_2 < 1$. Suppose the initial condition is Ψ_{M}. The two values of α are Ψ_{M} and Ψ_{m} corresponding to the given Ψ_{M}. If, next, we choose the initial condition as the Ψ_{m} corresponding to the Ψ_{M} chosen before, we will, once again, get two values of α, which are the same as before. Our objective now will be to derive, in general, the relationships between these four solutions. To achieve this, we will consider the two values of α, $\alpha_1 \triangleq \Psi_M(>1)$ and $\alpha_2 \triangleq \Psi_m(<1)$ irrespective of whether the initial condition is Ψ_M or Ψ_m.

For $\alpha = \alpha_1(>1)$ we obtain, from (41),

$$\pm\sqrt{-a_1}\int_R^{R_1} \mathrm{d}R = \int_\phi^{\beta_1} \frac{\mathrm{d}\phi}{\sqrt{-\tilde{p}_1(\phi)}} \tag{51}$$

where

$$\beta_1 = \beta|_{\alpha=\alpha_1=\Psi_M} = \frac{1}{\Psi_{\mathrm{ext}} - \alpha}\bigg|_{\alpha=\alpha_1=\Psi_M} = \frac{1}{\Psi_m - \Psi_M} \quad \text{using (44)} \tag{52}$$

$$\tilde{p}_1(\phi) = \frac{p(\phi)|_{\alpha=\alpha_1=\Psi_M}}{a_1} = \phi^3 + \frac{b_1}{a_1}\phi^2 + \frac{c_1}{a_1}\phi + \frac{d_1}{a_1} \quad \text{using (41)} \tag{53}$$

where a_1, b_1, c_1 and d_1 are defined by (42) with $\alpha = \alpha_1 = \Psi_M$, and R_1 denotes an integration constant which will later be determined from the initial condition. Now (51) can be re-expressed as (Abramowitz and Stegun 1970)

$$\pm\sqrt{-a_1}\lambda_1(R - R_1) = \lambda_1 \int_\phi^{\beta_1} \frac{\mathrm{d}\phi}{\sqrt{-\tilde{p}_1(\phi)}} \tag{54}$$

$$= F(\Phi_1/90° - \delta_1)$$

$$= \int_0^{\Phi_1} (1 - \cos^2\delta_1 \sin^2\theta)^{-1/2}\,\mathrm{d}\theta \tag{55}$$

where

$$\lambda_1^2 = [\tilde{p}_1'(\beta_1)]^{1/2} \tag{56}$$

and the parameter

$$\cos^2\delta_1 \triangleq 1 - m_1 = \tfrac{1}{2} + \tfrac{1}{8}\tilde{p}_1''(\beta_1)/[\tilde{p}_1'(\beta_1)]^{1/2}. \tag{57}$$

In (56) and (57), the primes denote differentiation with respect to ϕ. From (55),

$$\cos\Phi_1 \triangleq \frac{\lambda_1^2 - (\beta_1 - \phi)}{\lambda_1^2 + (\lambda_1 - \phi)}$$

$$= \mathrm{cn}[\sqrt{-a_1}\lambda_1(R - R_1)]. \tag{58}$$

Then, using the transformations (40) and (22), and the definition of a_1 in (42a),

$$\Psi_1(R) = \alpha_1 + \frac{1 + \mathrm{cn}[(2\alpha_1^3 - 2\alpha_1^2)^{1/2}\lambda_1(R - R_1)]}{(\beta_1 - \lambda_1^2) + (\beta_1 + \lambda_1^2)\,\mathrm{cn}[(2\alpha_1^3 - 2\alpha_1^2)^{1/2}\lambda_1(R - R_1)]}. \tag{59}$$

The period of the cn function, and hence of Ψ_1 is given as

$$\Lambda_1 = \frac{4\bar{K}(1 - m_1)}{(2\alpha_1^3 - 2\alpha_1^2\lambda_1)^{1/2}} \tag{60}$$

where

$$\bar{K}(\mu) \triangleq \int_0^{\pi/2} (1 - \mu\sin^2\theta)^{-1/2}\,\mathrm{d}\theta. \tag{61}$$

Similarly, for $\alpha=\alpha_2$ (<1) we obtain, from (41),

$$\pm\sqrt{a_2}\int_{R_2}^{R}\mathrm{d}R=\int_{\beta_2}^{\phi}\frac{\mathrm{d}\phi}{\sqrt{\tilde{p}_2(\phi)}} \tag{62}$$

where

$$\beta_2=\beta|_{\alpha=\alpha_2=\Psi_m}=\frac{1}{\Psi_{ext}-\alpha}\bigg|_{\alpha=\alpha_2=\Psi_m}=\frac{1}{\Psi_M-\Psi_m}=-\beta_1 \qquad \text{using (44)} \tag{63}$$

$$\tilde{p}_2(\phi)=\frac{p(\phi)|_{\alpha=\alpha_2=\Psi_m}}{a_2}=\phi^3+\frac{b_2}{a_2}\phi^2+\frac{c_2}{a_2}\phi+\frac{d_2}{a_2} \qquad \text{using (41)} \tag{64}$$

where a_2, b_2, c_2 and d_2 are defined by (42) with $\alpha=\alpha_2=\Psi_m$, and R_2 denotes an integration constant. The final solution for $\Psi(R)$ in this case becomes (Abramowitz and Stegun 1970):

$$\Psi_2(R)=\alpha_2+\frac{1+\mathrm{cn}[(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2(R-R_2)]}{(\beta_2+\lambda_2^2)+(\beta_2-\lambda_2^2)\,\mathrm{cn}[(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2(R-R_2)]} \tag{65}$$

where

$$\lambda_2^2=[\tilde{p}_2'(\beta_2)]^{1/2} \tag{66}$$

the parameter m_2 is given by

$$m_2=\tfrac{1}{2}-\tfrac{1}{8}\tilde{p}_2''(\beta_2)/[\tilde{p}_2'(\beta_2)]^{1/2} \tag{67}$$

and the period Λ_2 is given by

$$\Lambda_2=\frac{4\bar{K}(m_2)}{(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2} \tag{68}$$

with $\bar{K}(\cdot)$ defined in (61).

To establish the relationships between (59) and (65) as well as between the λ, m and Λ, we will state and prove the following lemmas.

Lemma 3.4.

$$(2\alpha_1^3-2\alpha_1^2)^{1/2}\lambda_1=(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2. \tag{69}$$

Proof. Using (56) we obtain

$$\begin{aligned}(2\alpha_1^3-2\alpha_1^2)^{1/2}\lambda_1&=\{2\Psi_M^3-2\Psi_M^2\}^{1/4}\\&\quad\times\frac{[2\Psi_M^3-\frac{8}{3}\Psi_M^2+2\Psi_M^2\Psi_m-\frac{8}{3}\Psi_M\Psi_m+2\Psi_M\Psi_m^2-\frac{2}{3}\Psi_m^2]^{1/4}}{((\Psi_m-\Psi_M)^2)^{1/4}}\\&\triangleq\frac{\{N_{11}\}^{1/4}[N_{12}]^{1/4}}{(\Psi_M-\Psi_m)^{1/2}}.\end{aligned} \tag{70}$$

Similarly,

$$\begin{aligned}(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2&=\{2\Psi_m^2-2\Psi_m^3\}^{1/4}\\&\quad\times\frac{[-(2\Psi_m^3-\frac{8}{3}\Psi_m^2+2\Psi_m^2\Psi_M-\frac{8}{3}\Psi_m\Psi_M+2\Psi_m\Psi_M^2-\frac{2}{3}\Psi_M^2)]^{1/4}}{((\Psi_M-\Psi_m)^2)^{1/4}}\\&\triangleq\frac{\{N_{21}\}^{1/4}[N_{22}]^{1/4}}{(\Psi_M-\Psi_m)^{1/2}}.\end{aligned} \tag{71}$$

We now need only

$$N_{12}=N_{21} \tag{72}$$

and

$$N_{11}=N_{22} \tag{73}$$

which are easy to check using (43).

Lemma 3.5.

$$1-m_1=m_2. \tag{74}$$

Proof. Proving (74) is equivalent to showing (see (57) and (67)) that

$$\frac{\tilde{p}_1''(\beta_1)}{(\tilde{p}_1'(\beta_1))^{1/2}}=-\frac{\tilde{p}_2''(\beta_2)}{(\tilde{p}_2'(\beta_2))^{1/2}}. \tag{75}$$

Now, after some algebra,

$$\text{LHS}=-\frac{(6\Psi_M^3-4\Psi_M\Psi_m-8\Psi_M^2+6\Psi_M^2\Psi_m)}{[N_{11}N_{12}]^{1/2}} \tag{76}$$

and

$$\text{RHS}=\frac{(6\Psi_m^3-4\Psi_m\Psi_M-8\Psi_m^2+6\Psi_m^2\Psi_M)}{[N_{22}N_{21}]^{1/2}}. \tag{77}$$

In expressions (76) and (77), the denominators are equal (using (72) and (73)) and the numerators are equal too (by virtue of (43)) and the lemma is proved.

Lemma 3.6.

$$\Lambda_1=\Lambda_2. \tag{78}$$

This follows directly from (69), (74) and the definition for $\bar{K}$ in (61).

We now proceed to impose initial conditions on the solutions (59) and (65) to find the constants R_1 and R_2 and thus construct the final solutions.

From (59), setting the initial condition as Ψ_M makes

$$(2\alpha_1^3-2\alpha_1^2)^{1/2}\lambda_1 R_1=2\bar{K}(1-m_1) \qquad [=2\bar{K}(m_2)] \tag{79}$$

so that the corresponding solution for $\Psi_1(R)$, which we shall call $\Psi_{1M}(R)$, is

$$\Psi_{1M}(R)=\Psi_M+\frac{1-\text{cn}[(2\alpha_1^3-2\alpha_1^2)^{1/2}\lambda_1 R]}{(\beta_1-\lambda_1^2)-(\beta_1+\lambda_1^2)\,\text{cn}[(2\alpha_1^3-2\alpha_1^2)^{1/2}\lambda_1 R]}. \tag{80}$$

If we set the initial condition in (59) as Ψ_m, it is readily verified that $R_1=0$, for the RHS of (59) becomes equal to $\Psi_M+1/\beta_1=\Psi_m$ (using (52)), which is the LHS. The corresponding solution, which we shall call $\Psi_{1m}(R)$, is:

$$\Psi_{1m}(R)=\Psi_M+\frac{1+\text{cn}[(2\alpha_1^3-2\alpha_1^2)^{1/2}\lambda_1 R]}{(\beta_1-\lambda_1^2)+(\beta_1+\lambda_1^2)\,\text{cn}[(2\alpha_1^3-2\alpha_1^2)^{1/2}\lambda_1 R]}. \tag{81}$$

It is easy to see that there is a half-period shift $(=2\bar{K}(1-m_1))$ between Ψ_{1M} and Ψ_{1m}, as is to be expected.

Similarly, starting from (65) and setting the initial conditions as Ψ_M and Ψ_m gives

$$\Psi_{2M}(R)=\Psi_m+\frac{1+\mathrm{cn}[(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2 R]}{(\beta_2+\lambda_2^2)+(\beta_2-\lambda_2^2)\,\mathrm{cn}[(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2 R]} \tag{82}$$

and

$$\Psi_{2m}(R)=\Psi_m+\frac{1-\mathrm{cn}[(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2 R]}{(\beta_2+\lambda_2^2)-(\beta_2-\lambda_2^2)\,\mathrm{cn}[(2\alpha_2^2-\alpha_2^3)^{1/2}\lambda_2 R]} \tag{83}$$

respectively. Once again, there is a half-period shift $(=2\bar{K}(m_2)=2\bar{K}(1-m_1))$ between the two solutions.

Finally, we have to establish the relationships between Ψ_{1M} and Ψ_{2M} and between Ψ_{1m} and Ψ_{2m}. We will deduce the first, the second follows along similar lines. First, we rewrite (80) and (82) as follows:

$$\Psi_{1M}(R)=\left(1-\frac{\Psi_M(\beta_1+\lambda_1^2)+1}{\Psi_M(\beta_1-\lambda_1^2)+1}\,\mathrm{cn}(\cdot)R\right)\times\left(\frac{\beta_1-\lambda_1^2}{\Psi_M(\beta_1-\lambda_1^2)+1}-\frac{\beta_1+\lambda_1^2}{\Psi_M(\beta_1-\lambda_1^2)+1}\,\mathrm{cn}(\cdot)R\right)^{-1} \tag{84}$$

$$\Psi_{2M}(R)=\left(1+\frac{\Psi_m(\beta_2-\lambda_2^2)+1}{\Psi_m(\beta_2+\lambda_2^2)+1}\,\mathrm{cn}(\cdot)R\right)\times\left(\frac{\beta_2+\lambda_2^2}{\Psi_m(\beta_2+\lambda_2^2)+1}+\frac{\beta_2-\lambda_2^2}{\Psi_m(\beta_2+\lambda_2^2)+1}\,\mathrm{cn}(\cdot)R\right)^{-1} \tag{85}$$

where

$$(\cdot)=(2\alpha_1^3-2\alpha_1^2)^{1/2}\lambda_1=(2\alpha_2^2-2\alpha_2^3)^{1/2}\lambda_2. \tag{86}$$

Lemma 3.7. We make the three propositions that:

$$-\frac{\Psi_M(\beta_1+\lambda_1^2)+1}{\Psi_M(\beta_1-\lambda_1^2)+1}=\frac{\Psi_m(\beta_2-\lambda_2^2)+1}{\Psi_m(\beta_2+\lambda_2^2)+1} \tag{87}$$

$$\frac{\beta_1-\lambda_1^2}{\Psi_M(\beta_1-\lambda_1^2)+1}=\frac{\beta_2+\lambda_2^2}{\Psi_m(\beta_2+\lambda_2^2)+1} \tag{88}$$

$$-\frac{(\beta_1+\lambda_1^2)}{\Psi_M(\beta_1-\lambda_1^2)+1}=\frac{\beta_2-\lambda_2^2}{\Psi_m(\beta_2+\lambda_2^2)+1}. \tag{89}$$

Proof. Note that, from (70) and (71),

$$\lambda_1^2=(N_{12}/N_{11})^{1/2}\beta_2 \tag{90}$$

and

$$\lambda_2^2=-(N_{22}/N_{21})^{1/2}\beta_1. \tag{91}$$

On using (72) and (73), it follows that

$$\lambda_1^2\lambda_2^2=-\beta_1\beta_2. \tag{92}$$

Now using the definitions of β_1 and β_2 (from (63)) equations (87)-(89) all reduce to (92) and the lemma is proved.

On the basis of the above lemma, we conclude that the solutions $\Psi_{1M}(R)$ and $\Psi_{2M}(R)$ are identical to each other. In a similar way, it follows that $\Psi_{1m}(R)=\Psi_{2m}(R)$.

This means that corresponding to a given initial condition, the final solution is the same, irrespective of which value of α we choose. Also, if a second initial condition equal to the other extremum is specified, the solution has a half-period shift with respect to the first solution.

We will now illustrate the theory advanced thus far by means of some examples.

(i) $\Psi_0=2$; $\Psi'(0)=0$.

Remark that since $\Psi_0>\frac{4}{3}$, this corresponds to the maximum value Ψ_M where the minimum value is expected to be negative. For this case, the two values of α are $\alpha_1=2.0$ and α_2 $(=\Psi_m)=-1.270$. The parameter $(1-m_1)$ equals 0.571. The period, calculated analytically, becomes equal to 5.015. The analytic solution is

$$\Psi_{1M}(R)=\Psi_{2M}(R)=\frac{1+4.947\,\mathrm{cn}(1.530\,R)}{3.041-0.067\,\mathrm{cn}(1.530\,R)}. \tag{93}$$

Equation (93) was plotted by computer by employing tables of cn functions (Fettis and Caslin 1965) and suitable polynomial interpolations to take care of the values not listed. Results are shown in figure 1(*a*) and agree remarkably well with the results from numerical solutions of (19) for the same initial conditions. Changing the initial condition to $\Psi_0=\Psi_m$ produces an analytic solution which when plotted (but not shown

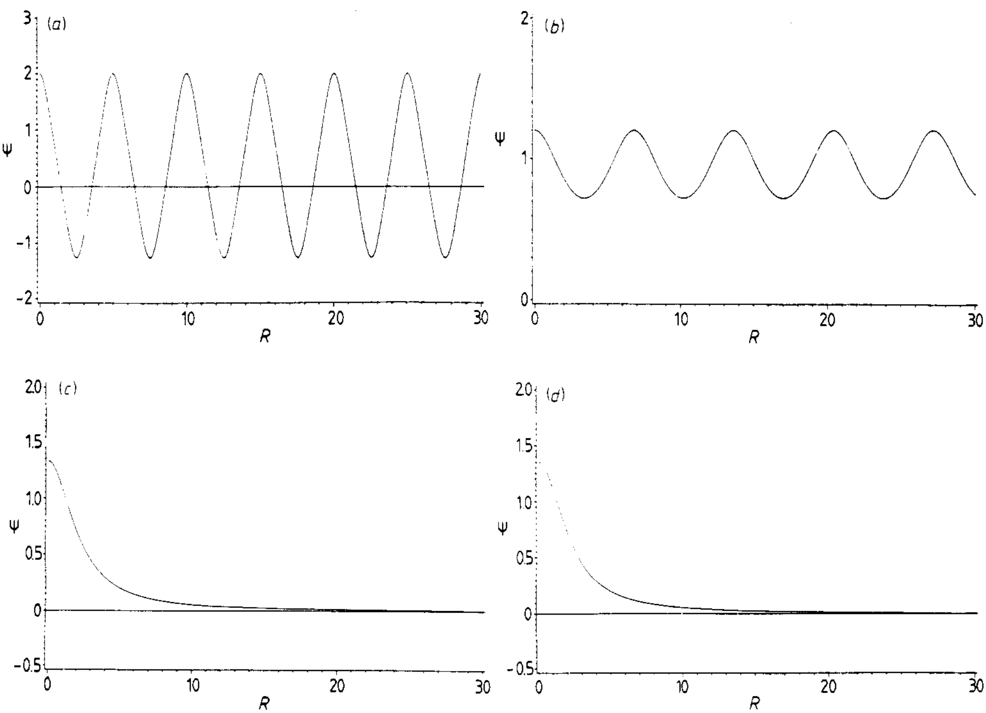

Figure 1. Plots of analytic solutions of equation (19) with initial conditions $\Psi(R=0)=(a)$ 2, (b) 1.2, (c) $\frac{4}{3}$ and $\Psi'(R=0)=0$, and (d) plot of numerical solution of equation (19) modified by a fourth-order non-linearity having initial conditions $\Psi(R=0)=1.35$, $\Psi'(R=0)=0$ with $A_4=0.011\,44$.

here) depicts the half-period shift with respect to the plot in figure 1(a). Once again, this agrees remarkably well with results from numerical solutions of (19).

(ii) $\Psi_0 = 1.2$; $\Psi'(0) = 0$.

Since $1 < \Psi_0 < \frac{4}{3}$, this corresponds, once again, to the maximum value Ψ_M, but where the minimum value is between 0 and 1. The two values of α equal 1.2 and 0.723. The parameter $(1 - m_1)$ equals 0.0035, and the period is 6.794. The analytic solution is plotted in figure 1(b) and once again exhibits the same properties as discussed for example (i).

For comparison, a plot of the algebraic solution, corresponding to $\Psi_0 = \frac{4}{3}$ and $\Psi'(0) = 0$ is given in figure 1(c).

In passing, it may be mentioned that for initial conditions slightly different from $\frac{4}{3}$, the solution to (19) may be made to tend to zero as R tends to infinity by incorporating a small amount of 'saturating' fourth-order non-linearity. This is illustrated in figure 1(d) by numerically solving the ODE for $\Psi_0 = 1.35$ and $A_4 \simeq 0.011\,44$, where A_4 is the coefficient of the Ψ^4 term which now needs to be added to the RHS of (19).

4. Higher-order modes—numerical solutions

It is clear from the discussion in the last section that for $\tilde{n} = 0$, (18) has only one solution (or mode) that tends to zero as R tends to infinity. The value of this function at $R = 0$ is $\frac{4}{3}$; for all other initial values the solutions are periodic in nature.

However, for $\tilde{n} = 1$ and 2 corresponding to circularly and spherically symmetric solutions, respectively, we know from non-linear optics (where the RHS of (8) is $\Psi - \Psi^3$) (Haus 1966) that there exist higher-order modes for which the solutions go to zero as R tends to infinity for a discrete set of initial conditions. For the non-linear optics case, these solutions can be obtained through numerical solution of the differential equation. Is the same true for (8) too? We have found the answer in the affirmative, as will be shown below.

Numerical solutions to (8) for initial conditions $\psi_0 = 2$, 3.1035 and 3.8530 and for $\tilde{n} = 1$ are shown in figure 2(a). These solutions, which may be characterised by mode numbers $m = 0$, 1 and 2, respectively, decay to zero as R tends to infinity. The curve for the initial condition equal to 2 corresponds to the algebraic solution, already discussed in § 2. For other initial conditions, for instance, 2.02, the solution does not decay to zero; rather, it exhibits oscillations with decreasing amplitudes as shown in figure 2(b). The period of the oscillations starts out at approximately 8.4 and tends to decrease to a limiting value. This makes sense, since at the onset of oscillations, the peak amplitude is around 1.3, which may roughly be considered a perturbation around the steady state value of 1. In fact, a perturbation analysis of (8) with $\tilde{n} = 1$ yields the equation

$$\frac{d^2\Delta\Psi}{dR^2} + \frac{1}{R}\frac{d\Delta\Psi}{dR} + \Delta\Psi = 0 \tag{94}$$

where $\Delta\Psi = \Psi - 1$. This is Bessel's equation of order zero. As R tends to infinity, this solution should behave as $\cos(R - \pi/4)/\sqrt{R}$ (Pipes and Harvill 1970) and hence should be periodic with a period 2π. (In fact, setting an initial condition equal to 1.3 verifies this from numerical simulation.) We speculate, therefore, that for arbitrary initial conditions, the solutions should be asymptotically stable.

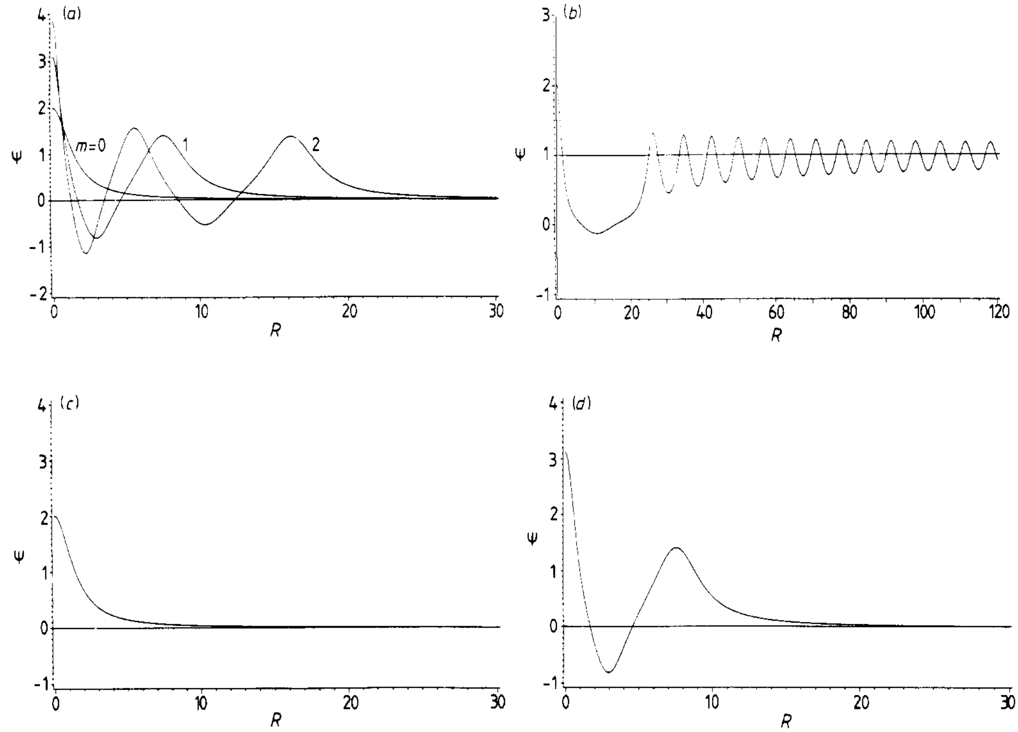

Figure 2. Numerical solutions of equation (8) for $\tilde{n} = 1$. (a) Higher-order modes with mode numbers 0, 1 and 2 corresponding to initial conditions $\Psi(R=0) = 2$, 3.1035 and 3.8530, respectively, and $\Psi'(R=0) = 0$. The modes have the characteristic property of tending to zero as $R \to \infty$. (b) With initial conditions $\Psi(R=0) = 2.02$ and $\Psi'(R=0) = 0$. (c) Modified by a fourth-order non-linearity having initial conditions $\Psi(R=0) = 2.02$ and $\Psi'(R=0) = 0$ with $A_4 = 0.004\,55$. (d) Modified by a fourth-order non-linearity having initial conditions $\Psi(R=0) = 3.12$ and $\Psi'(R=0) = 0$ with $A_4 = 0.0051$.

Similar to the $\tilde{n} = 0$ case, solutions of (8), corresponding to initial conditions slightly different from the values yielding solutions that finally go to zero, may also be made to tend to zero as R tends to infinity by incorporating a small amount of 'saturating' fourth-order non-linearity in the system. This is illustrated in figure 2(c) for $\Psi_0 = 2.01$ and $A_4 \simeq 0.004\,55$ where A_4 is the coefficient of the Ψ^4 term which now modifies (8). A plot for $\Psi_0 = 3.12$ and $A_4 = 0.0051$ has also been added (see figure 2(d)) to illustrate the same for an initial condition close to that for the solution with mode number $m = 1$.

5. Reduction of other Klein–Gordon systems to the Klein–Gordon equation without dispersion

Consider a more general Klein-Gordon equation having a dispersion term, quadratic and cubic non-linearities, and modelling a system impressed by a constant force F (Lal 1985, 1986):

$$\partial^2\psi/\partial t^2 - c_0^2\nabla^2\psi = A_1\psi + A_2\psi^2 + A_3\psi^3 + F. \tag{95}$$

Following exactly the same procedure employing in deriving (6) from (2), we get, from

(95), the ODE

$$\frac{d^2\tilde{\tilde{\psi}}}{dr^2}+\frac{\tilde{n}}{r}\frac{d\tilde{\tilde{\psi}}}{dr}=-A_1\tilde{\tilde{\psi}}-A_2\tilde{\tilde{\psi}}^2-A_3\tilde{\tilde{\psi}}^3-F. \tag{96}$$

Setting

$$\tilde{\tilde{\psi}}=\bar{\psi}+\bar{c} \tag{97}$$

in (96) transforms it to

$$\frac{d^2\bar{\psi}}{dr^2}+\frac{\tilde{n}}{r}\frac{d\bar{\psi}}{dr}=-\tilde{A}_2\bar{\psi}^2-\tilde{A}_3\bar{\psi}^3 \tag{98}$$

with

$$\tilde{A}_2=A_2+3\bar{c}A_3 \tag{99}$$

$$\tilde{A}_3=A_3 \tag{100}$$

provided

$$F+\bar{c}A_1+\bar{c}^2A_2+\bar{c}^3A_3=0 \tag{101}$$

and

$$A_1+2\bar{c}A_2+3\bar{c}^2A_3=0. \tag{102}$$

Finally, normalisation using

$$\Psi=-\frac{\tilde{A}_3}{\tilde{A}_2}\bar{\psi} \qquad R=\left(\frac{\tilde{A}_2^2}{\tilde{A}_3}\right)^{1/2}r \qquad \tilde{A}_3>0 \tag{103}$$

reduces (98) to

$$\frac{d^2\Psi}{dR^2}+\frac{\tilde{n}}{R}\frac{d\Psi}{dR}=\Psi^2-\Psi^3 \tag{104}$$

which is identical to (8). All the results derived in §§ 2, 3 and 4 can therefore be used.

Let us take a moment to reflect on the conditions that made possible the reduction to the form in (8). Note that if we impose the requirement that $\bar{c}=0$, then $F=0$ (from (101)) and $A_1=0$ (from (102)). Also, if $F=0$, it follows that there exists a possibility for algebraic solitary waves in a Klein-Gordon system with dispersion, and with quadratic and cubic non-linearities if the non-linear parameters satisfy the condition

$$A_2^2/A_3=4A_1. \tag{105}$$

The algebraic solitary waves do not, however, decay to zero as $R\to\infty$, rather, to

$$\bar{c}=-A_2/2A_3. \tag{106}$$

Finally, for a given constant force, algebraic solitary waves may exist, whether or not the system exhibits dispersion.

6. Conclusion

In conclusion, the non-linear Klein-Gordon equation without dispersion and with quadratic and cubic non-linearities has been studied in one and higher dimensions.

Algebraic solitary wave solutions have been deduced for the one-dimensional case, and for higher-dimensional cases exhibiting circular and spherical symmetry, corresponding to specific initial values in a moving frame of reference. For arbitrary initial values, it is shown that solutions are periodic in the one-dimensional case. In the higher-dimensional case, different modes, depending on the initial values, have been shown to exist. Any other initial condition is conjectured to yield solutions that are asymptotically stable. For both one- and higher-dimensional cases, solutions tending to zero with distance are shown to be achieved for initial conditions, close to the special set of initial values that exhibit the property, by incorporating a small amount of 'saturating' fourth-order non-linearity in the system. Finally, it is shown how a fairly general Klein-Gordon equation having dispersion and a forcing term as well as quadratic and cubic non-linearities may be reduced to the system discussed in the paper.

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